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LETTER TO THE EDITOR

Bifurcations in the periodic orbit sum: multiple windings

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Abstract. Periodic orbits that participate in a bifurcation contribute collectively to the periodic orbit sum for the quantum density of states. The contributions of multiple windings of isolated orbits are easily obtained from powers of the stability matrix, but it is generally hard to compose the actions that determine the contributions of higher windings of a bifurcation. Here we derive an approximate relation between the amplitude of the contributions of different windings for the saddle-centre bifurcation and the period-doubling bifurcation.

The periodic orbit sum is one of the main tools for the study of spectral fluctuations in quantum systems. Semiclassical periodic-orbit approximations have been derived for the limitting cases of chaotic or integrable systems, where the orbits are either isolated or appear in continuous families [1, 2]. Generic systems are harder to treat because of bifurcations in which periodic orbits coalesce as a parameter is varied. In the case of a single quantum map, one may hope to avoid bifurcations of short-period orbits, but for Hamiltonian systems the energy itself is such a parameter, leading to complex sequences of bifurcations. The joint contribution to the periodic orbit sum of orbits which participate in a bifurcation was first derived by Ozorio de Almeida and Hannay [3], and subsequently refined by Schomerus and Sieber [4].

In spite of these advances, there is no doubt that the collective contribution of bifurcating orbits is much harder to evaluate than those of orbits whose actions differ by considerably more than $\hbar$ and can thus be considered to be isolated. This is specially true in the case of higher windings, i.e. multiple iterations of the Poincaré map transverse to the periodic orbits; given the linearization of the map around its isolated fixed point (where the periodic orbit intersects the Poincaré section),

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \mathsf{M} \begin{pmatrix} Q \\ P \end{pmatrix} \tag{1}$$

the amplitude A_m of the contribution of the mth winding of the periodic orbit is easily determined since

$$A_m \propto |\det(\mathsf{M}^m - 1)|^{\frac{1}{2}}. \tag{2}$$

Thus, there is no further effort of retrieving classical information, once the first iteration is dealt with. As the phase of the contribution is merely multiplied by m, it also becomes possible to sum over a sequence of windings of a single orbit.

In contrast, the contribution of orbits undergoing a bifurcation has amplitude

$$A_m = (2\pi \mathrm{i}\hbar)^{-\frac{1}{2}} \int \mathrm{d}P\, \mathrm{d}Q \left| \frac{\partial^2 S_m}{\partial P \partial Q} \right|^{\frac{1}{2}} \exp\left\{ \frac{\mathrm{i}}{\hbar} [S_m(P, Q, \epsilon) - PQ] \right\} \tag{3}$$

where $S_m(P, Q', \epsilon)$ is the normal form for the particular bifurcation in the classification of Meyer [5] and Bruno [6] and the Poincaré map is implicitly defined by

$$Q' = \frac{\partial S_m}{\partial P'} \qquad P = \frac{\partial S_m}{\partial Q}. \tag{4}$$

Beyond the unavoidable difficulty that the map cannot be reduced to its linear approximation, we need to work out the relation of S_m to S_1 for each winding. It is important to note that here '1' refers to the first winding of the Poincaré map at which the bifurcation manifests itself, which may occur after several windings of the central periodic orbit. Because all windings have a Poincaré map with the same fixed points (for a fixed parameter ϵ), they will be described by the same form of catastrophe integral (3), though with different amplitudes and control variables.

The purpose of this letter is to relate the contributions of the various windings for two fundamental bifurcations. Whereas isolated orbits can be described by quadratic generating functions,

$$S(P, Q') = PQ' - \epsilon Q' + Q'^3 + P^2 \tag{5}$$

is the normal form for the *saddle-centre bifurcation* and

$$S(P, Q') = PQ' + \epsilon Q'^2 + Q'^4 + P^2 \tag{6}$$

for the *period-doubling bifurcation*. In the first case, a stable and an unstable orbit coalesce and disappear as $\epsilon \to 0$; in the second, a stable orbit of twice the period arises as the central orbit loses its stability.

To evaluate further iterations, we derive the explicit maps corresponding to (5) and (6). In the case of the saddle-centre bifurcation,

$$Q' = Q - 2p \qquad P' = P - \epsilon + 3Q'^2. \tag{7}$$

For a small bifurcation parameter ϵ, the fixed points lie close to the origin. Thus, expanding to the lowest order in Q, P and ϵ, we obtain the mth iteration as

$$Q^{(m)} \approx Q - 2mP \qquad \text{and} \qquad P^{(m)} \approx P - m\epsilon + 3m(Q^{(m)})^2. \tag{8}$$

The corresponding generating function is hence

$$S(P, Q^{(m)}) \approx PQ^{(m)} - m\epsilon Q^{(m)} + m(Q^{(m)})^3 + mP^2. \tag{9}$$

The corresponding deduction for the period-doubling bifurcation leads to the map

$$Q^{(m)} \approx Q - 2mP \qquad P^{(m)} \approx P + 2m\epsilon Q^{(m)} + 4m(Q^{(m)})^3 \tag{10}$$

generated by

$$S(P, Q^{(m)}) \approx PQ^{(m)} + m\epsilon(Q^{(m)})^2 + m(Q^{(m)})^4 + mP^2. \tag{11}$$

We can now obtain the approximate amplitude for each winding by inserting (9) or (11) into (3). However, the picture becomes clearer if we use instead the position generating functions

$$S(Q, Q^{(m)}) \approx -\frac{1}{2m}(Q^{(m)} - Q)^2 - m\epsilon Q^{(m)} + m(Q^{(m)})^3 \tag{12}$$

(for the saddle-centre bifurcation) and

$$S(Q, Q^{(m)}) \approx -\frac{1}{2m}(Q^{(m)} - Q)^2 + m\epsilon(Q^{(m)})^2 + m(Q^{(m)})^4 \tag{13}$$

(for the period-doubling bifurcation) in the single integral

$$A_m = \int \mathrm{d}Q \left|\frac{\partial^2 S_m}{\partial Q \partial Q^{(m)}}\right|^{\frac{1}{2}} \exp\left\{\frac{\mathrm{i}}{\hbar} S_m(Q, Q)\right\}. \tag{14}$$

Though this form fails for the identity transformation, it need not have been discarded in [3]. Instead, it results from the evaluation of the Gaussian integral in P in (3), since $S_m(Q, Q)$ is just the Legendre transform of $S_m(P, Q)$. The result is that the amplitudes now depend on single Airy [7] and Pearcey integrals [8, 9] where the phase is merely multiplied by the winding number m. Note, however, that the amplitude is divided by $m^{\frac{1}{2}}$, indeed

$$A_m \approx m^{-\frac{1}{2}} \int \mathrm{d}Q \left|\frac{\partial^2 S_1}{\partial Q \partial Q'}\right|^{\frac{1}{2}} \exp\left\{\frac{\mathrm{i}}{\hbar} m S_1(Q, Q)\right\}. \tag{15}$$

Equation (15) is the main result of this letter. The contributions to the periodic-orbit sum will not be as accurate as the full uniform approximation of [4], but we can now understand the relative contributions of the successive windings. Indeed, we obtain,

$$A_m(\epsilon, \hbar) \approx \hbar^{\frac{1}{3}} m^{-\frac{5}{6}} A_1\left(\epsilon\left(\frac{m}{\hbar}\right)^{\frac{2}{3}}, 1\right) \tag{16}$$

for the saddle-centre bifurcation and

$$A_m(\epsilon, \hbar) \approx \hbar^{\frac{1}{4}} m^{-\frac{3}{4}} A_1\left(\epsilon\left(\frac{m}{\hbar}\right)^{\frac{3}{4}}, 1\right) \tag{17}$$

for the period-doubling bifurcation. In both cases, the higher windings behave effectively as a reduction of Planck's constant. The result is an apparent increase of the bifurcation parameter. For sufficiently large windings, the periodic orbits can be treated as independent even when the lower windings must be considered collectively. This agrees with the general criterion that it is the action difference between the periodic orbits (multiplied by their winding number) that distinguishes between the two regimes. Now we find a simple rule for evaluating the contributions of many windings from the normal form of the first winding, even in the nonlinear regime.

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